For each town i, we can slice its days into several disjoint blocks: $[1, d_{i,1}], [d_{i,1}+1, d_{i,2}], ..., [d_{i,k}+1, d_{i,k+1}], ...$, where $d_{i,k} = min\{l | \sum_{j=1}^{l} g(i,j) \geq k\}$. For readers' better understanding, we give a simple example here. For instance, if a town's garbage production for each day is {Day 1:0.9, Day 2:0.1, Day 3:0.3, Day 4:0.6, Day 5:0.9, Day 6:0.5, Day 7:0.4}, then the corresponding blocks are {[1,2],[3,5],[6]}.

We can show that, if we can find a schedule such that the car can visit every single block for each town, then for any town, the interval amount between two visits is less than 2, which means the schedule is a feasible solution. For an arbitrary town i, we consider its arbitrary two nearby blocks, namely the k-th block and the k+1-th block.

- If $k = 1$, and we denotes the amount of produced garbage between two visits in these two blocks is S, then according to the definition of our block construction, we set $d_{(i,k+1)}$ to be the minimum number of days that the sum of all garbage produced by this town until today exceeds $k+1$, therefore, the sum of all garbage until any day before $d_(i, k+1)$ must be less than $k+1$. Then we can have $S \leq \sum_{j=2}^{d_{i,k+1}-1} g(i,j) \leq \sum_{j=1}^{d_{i,k+1}-1} g(i,j) < k+1 = 2$.
- If $k \geq 2$, then according to the definition of our block construction, we can have $\sum_{j=1}^{d_{i,k-1}} g(i,j) \geq k-1$ and $\sum_{j=1}^{d_{i,k+1}-1} g(i,j) < k+1$. Same as above, we denotes the amount of produced garbage between two visits in these two blocks is S. Then $S \leq \sum_{j=d_{i,k-1}+2}^{d_{i,k+1}-1} g(i,j) \leq \sum_{j=d_{i,k-1}+1}^{d_{i,k+1}-1} g(i,j) = \sum_{j=1}^{d_{i,k+1}-1} g(i,j) - \sum_{j=1}^{d_{i,k-1}} g(i,j) < (k+1) - (k-1) = 2$.

Then, we can transform the original problem into a bipartite matching problem, in which the left vertices set X are consist of blocks and each block are a consecutive series of days, right vertices set Y are consist of days and the edges are from each block to its corresponding days, that means $E = \{(u,v) | u \in X, v \in Y, v \in u\}$. (For example, if a town has a block [3,6], then its edges are from this block to Day 3,4,5,6). If one town produces x garbage in total (starting from and including day 1), it's easy to see that this town will have $\lfloor x \rfloor$ blocks in total. Because every day's total garbage is less than or equal 1, then the total number of blocks is less than or equal number of days, which means the $|X|$ (number of left vertices) is less than or equal $|Y|$ (the number of days). So, if we can prove for such matching problem, each $W \subseteq X$ satisfies $|\Gamma(W)| \geq |W|$ (the notations here are same as the ones in the lecture notes), then according to Hall's Theorem, we can get that, there exists a perfect matching for X.

So let's consider for any arbitrary $W \subseteq X$ and its corresponding $\Gamma(W)$. Here we denotes $|W| = m$ and $|\Gamma(W)| = n$.

- If $\Gamma(W)$ is a sequence of continuous n days starting from the k-th day and ending with the (k+n-1)-th day, namely the interval [k,k+n-1].
  - If $k = 1$, it's easy to see that the total amount of garbage in [1,n] is less than or equal n, thus the total amount of blocks that belong to this interval is also less than or equal n since the accumulating garbage until $i_{th}$ block must be greater or equal to $i$. So $m \leq n$.

- If $k \geq 2$, we denotes these blocks belong to p towns in total. Then these towns form such a set $\{a_1, a_2, ...a_p\}$. For each town $a_i$, we denotes the first block belongs to $W$(or say, whose corresponding days are the earliest) is the $b_i$-th block, namely, $b_i = min\{l|[d_{a_i,l-1}+1, d_{a_i,l}] \in W\}$. We also denotes each town $a_i$ contains $c_i$ blocks in total that belongs to W and obviously $\sum_{i=1}^{p} c_i = m$. Then according to the property of our block construction, for each town $a_i$, we can have $\sum_{j=1}^{k-2} g(a_i, j) < b_i - 1, \sum_{j=1}^{k+n-1} g(a_i, j) \geq b_i + (c_i - 1)$, thus $\sum_{j=k-1}^{k+n-1} g(a_i, j) > c_i$. Summing all the p towns, we can get $\sum_{i=1}^{p} c_i = m < \sum_{i=1}^{p} \sum_{j=k-1}^{k+n-1} g(a_i, j) \leq n+1$ as there are only $n+1$ days, the amount of garbage produced by all towns could be at most $n+1$, i.e. $m \leq n$.

- If $\Gamma(W)$ is non-continuous, since in our block construction method, each block's corresponding days are always continuous, so we can always divide $\Gamma(W)$ into several separate days-continuous disjoint subsets $\Gamma(W)_i$, s.t. $\bigcup_i \Gamma(W)_i = \Gamma(W)$ and $\forall i, j, \Gamma(W)_i \cap \Gamma(W)_j = \emptyset$ and also $\forall i \neq j, \forall$ day $a \in \Gamma(W)_i, b \in \Gamma(W)_j, |a-b| \geq 2$. (For example, if $\Gamma(W) = 1, 2, 3, 5, 6, 7, 9, 10$, we will divide it into 3 intervals: $[1, 3], [5, 7], [9, 10]$). Then we also separate $W$ into several partitions $W_i = \{x \in W | \Gamma(x) \subseteq \Gamma(W)_i\}$. Based on $\{\Gamma(W)_i\}$'s disjoint property and $W_i$'s definition, we can show that $\{W_i\}$ is also disjoint,i.e, $\forall i, j, W_i \cap W_j = \emptyset$: If $\exists x \in W_i$ and $x \in W_j$ and $i \neq j$, then $\Gamma(x) \subseteq \Gamma(W)_i \cap \Gamma(W)_j$, which is contradicted to $\{\Gamma(W)_i\}$ 's disjoint property. In the following part, we can show that $\Gamma(W_i) = \Gamma(W)_i$:

  - According to definition of $W_i$, $\forall y \in \Gamma(W_i), y \in \Gamma(W)_i$, so $\Gamma(W_i) \subseteq \Gamma(W)_i$.
  - Then we want to prove that $\Gamma(W)_i \subseteq \Gamma(W_i)$ by using contradiction. If we assume $\exists y \in \Gamma(W)_i$ and $y \notin \Gamma(W_i)$, but since $y \in \Gamma(W)$, then $\exists j \neq i,$, s.t. $y \in \Gamma(W_j)$, which leads to $y \in \Gamma(W)_j$. However, $\{\Gamma(W)_i\}$ are disjoint, so there's a contradiction. Then we can get $\forall y \in \Gamma(W)_i, y \in \Gamma(W_i)$, thus $\Gamma(W)_i \subseteq \Gamma(W_i)$.

  So based on the above proof, we can see that $\Gamma(W_i) = \Gamma(W)_i$. And because of the proof in the previous section, for each continuous interval, we have $|\Gamma(W)_i| \geq |W_i|$. And because these days intervals are disjoint, $|\Gamma(W)| = \sum_i |\Gamma(W)_i| \geq \sum_i |W_i| = |W|$,i.e, $m \leq n$.

Based on the above proof, we can guarantee that after building blocks using the way described, there exists a perfect matching for blocks such that every block can be matched with a individual day, such that for each town, the total amount of produced garbage between two visits is always less than 2, which means this is the feasible schedule we are looking for.

For the complexity, we first need to construct the bipartite graph described above, and this takes $O(n \times d)$ which $d$ is number of days nad $n$ is number of towns as we need to loop through all garbage produced by each town. As for the specific algorithm for finding such perfect matching, since $|X| \leq |Y|$, we can use Hopcroft and Karp algorithm to find the maximum matching on the bipartite graph we build. The time complexity is $O(m\sqrt{n})$ where m is the number of edges and n is the number of vertices. For our specific case, $n$ is basically the number of total days, $m$ is less than or equal $n^2$, so the time complexity in total is still polynomial.

End of proof.